AdS/QCD and the Holographic Light-Front Representation

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\[ \langle \zeta | H_{LF} | \phi \rangle = M^2 \langle \zeta | \phi \rangle \]

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1 Introduction

- Quark-gluon dynamics described by the Yang-Mills $SU(3)$ color QCD Lagrangian $\mathcal{L}_{\text{QCD}}$

\[
S_{\text{QCD}} = \int d^4x \mathcal{L}_{\text{QCD}}(x),
\]

\[
\mathcal{L}_{\text{QCD}} = -\frac{1}{4g^2} \text{Tr} \left( G^{\mu\nu} G_{\mu\nu} \right) + \sum_{f=1}^{n_f} i\bar{q}_f D_\mu \gamma^\mu q_f + \sum_{f=1}^{n_f} m_f \bar{q}_f q_f.
\]

- Dimensionless coupling renormalizable theory with asymptotic freedom and color confinement.

- Most challenging problem of strong interaction dynamics: How the fundamental constituents in the QCD Lagrangian appear in the physical spectrum as colorless states, mesons and baryons.

- Recent developments using the AdS/CFT correspondence between string states in AdS space and conformal field theories in physical space-time have renewed the hope of finding an analytical approximation to describe the confining dynamics of QCD, at least in its strongly coupling regime.
The Original Holographic Correspondence

- Holographic duality requires a higher dimensional warped space. Space with negative curvature and a 4-dim boundary: AdS$_5$.

- Original correspondence between $\mathcal{N} = 4$ SYM at large $N_C$ and the low energy supergravity approximation to Type IIB string on AdS$_5 \times S^5$ Maldacena, hep-th/9711200.

<table>
<thead>
<tr>
<th>Warped higher dim space</th>
<th>Conformal d = 4 spacetime boundary</th>
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<tbody>
<tr>
<td>Type IIB (AdS$_5 \times S^5$)</td>
<td>$\mathcal{N} = 4$ SYM (SO(4, 2) $\otimes$ SU(4))</td>
</tr>
<tr>
<td>?</td>
<td>QCD</td>
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</table>

- The group of conformal transformations $SO(4, 2)$ is also the group of isometries of AdS$_5$, and $S^5$ corresponds to the global $SU(4) \sim SO(6)$ group which rotates the particles in the SYM multiplet.

- Description of strongly coupled gauge theory using a dual gravity description!

- QCD is fundamentally different from SYM theories where all matter is in the adjoint rep of $SU(N_C)$, and is non-conformal. Is there a dual string theory to QCD?
Strongly Coupled QCD and AdS/CFT

• Effective gravity description of strongly coupled quasi-conformal QCD.

• Semi-classical correspondence as a first approximation to QCD (strongly coupled at all scales).

• Strings describe extended objects (no quarks). QCD degrees of freedom are pointlike particles: how can they be related? How can we map string states into partons?

• Precise mapping of string amplitudes to light-front wavefunctions of hadrons in the light-front for strongly coupled QCD in the conformal limit.

• Eigenvalues of normalizable modes inside AdS give the hadronic spectrum. AdS modes represent also the probability amplitude for distribution of quarks at a given scale.

• To each state of the gauge theory should correspond a normalized mode in AdS. The lowest stable mode should correspond to the lowest state of the QCD Hamiltonian.

• Non-normalizable modes are related to external currents: they probe the cavity interior. Also couple to boundary QCD interpolating operators.
• Isomorphism of $SO(4, 2)$ of conformal QCD with the group of isometries of AdS space 

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2).$$

• $x^\mu \rightarrow \lambda x^\mu$, $z \rightarrow \lambda z$, maps scale transformations into the holographic coordinate $z$.

• A distance $L_{AdS}$ shrinks by a warp factor as observed in Minkowski space ($dz = 0$):

$$L_{Minkowski} \sim \frac{z}{R} L_{AdS}.$$ 

• Different values of $z$ correspond to different scales at which the hadron is examined: AdS boundary at $z \rightarrow 0$ correspond to the $Q \rightarrow \infty$, UV zero separation limit.

• There is a maximum separation of quarks and a maximum value of $z$ at the IR boundary

• Truncated AdS/CFT model: cut-off at $z_0 = 1/\Lambda_{QCD}$ breaks conformal invariance and allows the introduction of the QCD scale (Hard-Wall Model) Polchinski and Strassler (2001).

• Smooth cutoff: introduction of a background dilaton field $\varphi(z)$ usual Regge dependence can be obtained (Soft-Wall Model) Karch, Katz, Son and Stephanov (2006).
Conformal QCD Window in Exclusive Processes

- In the semi-classical approximation to QCD with massless quarks and no quantum loops the $\beta$ function is zero and the approximate theory is scale and conformal invariant.

- Does $\alpha_s$ develop an IR fixed point? D-S Equation Alkofer, Fischer, LLanes-Estrada, Deur . . .

- Recent lattice simulations: evidence that $\alpha_s$ becomes constant and not small in the infrared Furui and Nakajima, hep-lat/0612009 (Green dashed curve: DSE).

- Phenomenological success of dimensional scaling laws for exclusive processes

$$d\sigma/dt \sim 1/s^{n-2}, \quad n = n_A + n_B + n_C + n_D,$$

implies QCD is a strongly coupled conformal theory at moderate but not asymptotic energies Brodsky and Farrar (1973); Matveev et al. (1973).
• Derivation of counting rules for gauge theories with mass gap dual to string theories in warped space (hard behavior instead of soft behavior characteristic of strings) Polchinski and Strassler (2001).

• Example: Dirac proton form factor: 

\[ F_1(Q^2) \sim \left[ \frac{1}{Q^2} \right]^{n-1}, \quad n = 3 \]

\[ Q^4 F_1^p(Q^2) \quad [\text{GeV}^4] \]

Finite Temperature Gauge Theory and AdS/CFT

- Perfect quark-gluon liquid observed at RHIC: strongly coupled state.

- Conjectured lower bound:
  \[ \frac{\eta}{S} \geq \frac{1}{4\pi} \]

- Quantum mechanical result derived from the AdS/CFT correspondence for all relativistic quantum field theories at finite temperature and zero chemical potential.

- Hydrodynamic gauge theory properties of AdS/CFT:
  Policastro, Son and Starinets (2001); Kovtun, Son and Starinets (2005).
2 The Holographic Correspondence and Interpolating Operators

Precise statement of duality between a gravity theory in AdS$_{d+1}$ and the strong coupling limit of a conformal field theory at the $z = 0$ boundary Gubser, Klebanov and Polyakov (1998); Witten (1998):

- $d+1$-dim gravity partition function for scalar field in AdS$_{d+1}$: $\Phi(x, z)$

$$Z_{\text{grav}}[\Phi(x, z)] = e^{iS_{\text{eff}}[\Phi]} = \int D[\Phi] e^{iS[\Phi]}.$$  

- $d$-dim generating functional in presence of external source $\Phi_0$

$$Z_{\text{QCD}}[\Phi_0(x)] = e^{iW_{\text{QCD}}[\Phi_0]} = \int D[\psi, \overline{\psi}, A] \exp \left\{ iS_{\text{QCD}} + i \int d^d x \Phi_0 \mathcal{O} \right\}.$$ 

with $\mathcal{O}$ a hadronic local interpolating operator ($\mathcal{O} = G_{\mu\nu}^{a}G^{a\mu\nu}, \overline{q}\gamma^5 q, \cdots$)

- Boundary condition:

$$Z_{\text{grav}}[\Phi(x, z = 0) = \Phi_0(x)] = Z_{\text{QCD}}[\Phi_0].$$

- Semi-Classical Approximation

$$W_{\text{QCD}}[\phi_0] = S_{\text{eff}}[\Phi(x, z)|_{z=0} = \Phi_0(x)]_{\text{on-shell}}.$$
• Near the boundary of $AdS_{d+1}$ space $z \to 0$:

$$\Phi(x, z) \to z^\Delta \Phi_+(x) + z^{d-\Delta} \Phi_-(x).$$

• $\Phi_-(x)$ is the boundary limit of non-normalizable mode (source): $\Phi_- = \Phi_0$

• $\Phi_+(x)$ is the boundary limit of the normalizable mode (physical states)

• Using the equations of motion AdS action reduces to a UV surface term

$$S_{eff} = \frac{R^{d-1}}{4} \lim_{z \to 0} \int d^d x \frac{1}{z^{d-1}} \Phi \partial_z \Phi,$$

• $S_{eff}$ is identified with the boundary functional $W_{CFT}$

$$\langle \mathcal{O} \rangle_{\Phi_0} = \frac{\delta W_{CFT}}{\delta \Phi_0} = \frac{\delta S_{eff}}{\delta \Phi_0} \sim \Phi_+(x),$$

Balasubramanian et. al. (1998), Klebanov and Witten (1999).

• Physical AdS modes $\Phi_P(x, z) \sim e^{-iP \cdot x} \Phi(z)$ are plane waves along the Poincaré coordinates with four-momentum $P^\mu$ and hadronic invariant mass states $P_\mu P^\mu = M^2$.

• For small-$z$ $\Phi(z) \sim z^\Delta$. The scaling dimension $\Delta$ of a normalizable string mode, is the same dimension of the interpolating operator $\mathcal{O}$ which creates a hadron out of the vacuum: $\langle P|\mathcal{O}|0\rangle \neq 0$. 
3 Bosonic Modes

• Conformal metric $x^\ell = (x^\mu, z)$:

$$ds^2 = g_{\ell m} dx^\ell dx^m = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2).$$

• Action for massive scalar modes on AdS$_{d+1}$:

$$S[\Phi] = \frac{1}{2} \int d^{d+1}x \sqrt{g} \frac{1}{2} \left[ g_{\ell m} \partial_\ell \Phi \partial_m \Phi - \mu^2 \Phi^2 \right],$$

with $\sqrt{g} \to (R/z)^{d+1}$ in the conformal limit.

• Equation of motion:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\ell} \left( \sqrt{g} g_{\ell m} \frac{\partial}{\partial x^m} \Phi \right) + \mu^2 \Phi = 0.$$
• Factor out dependence of string mode $\Phi_P(x, z)$ along $x^\mu$-coordinates

$$\Phi_P(x, z) = e^{-iP \cdot x} \Phi(z),$$

with $P_\mu P^\mu = M^2$.

• Find AdS equation of motion

$$[z^2 \partial_z^2 - (d - 1)z \partial_z + z^2 M^2 - (\mu R)^2] \Phi(z) = 0.$$

• Solution:

$$\Phi(x, z) = C z^{d \over 2} J_{\Delta - d \over 2} (zM),$$

• Conformal dimension $\Phi(z) \rightarrow z^{\Delta}$ as $z \rightarrow 0$,

$$\Delta = \frac{1}{2} \left( d + \sqrt{d^2 + 4\mu^2 R^2} \right).$$

• Normalization

$$R^{d-1} \int_0^{L_{QCD}^{-1}} \frac{dz}{z^{d-1}} \Phi^2_{S=0}(z) = 1.$$
Holographic Light-Front Representation (Hard Wall Model)

- We can represent the EOM in AdS space in light-front Lorentz invariant Hamiltonian form in physical 3+1 space-time at fixed LC time $\tau = t + z/c$

$$H_{LF}|\phi\rangle = M^2|\phi\rangle.$$ 

- Write the AdS metric in terms of light front coordinates $x^\pm = x^0 \pm x^3$

$$ds^2 = \frac{R^2}{z^2} \left( dx^+ dx^- - dx_\perp^2 - dz^2 \right).$$ 

- The AdS metric $ds^2$ is invariant if $x_\perp^2 \rightarrow \lambda^2 x_\perp^2$ and $z \rightarrow \lambda z$ at equal light-front time $\tau$.

Small $z$ related to small transverse dimensions!

- We can identify the light-front variable $\zeta$ in 3+1 space with the fifth dimension $z$ of AdS space, $\zeta = z$. The LF variable $\zeta$ represents the invariant transverse separation between pointlike constituents.

• Substitute in AdS EOM: 
  \[ \phi(\zeta) \sim \zeta^{-3/2} \Phi(\zeta), \quad (\mu R)^2 = -4 + \nu^2 \]
  \[ \left[ -\frac{d^2}{d\zeta^2} - \frac{1 - 4\nu^2}{4\zeta^2} \right] \phi(\zeta) = M^2 \phi(\zeta). \]

• Transverse impact holographic representation with light-front wavefunctions \( \phi(\zeta) = \langle \zeta | \phi \rangle \)
  \[ \langle \zeta | H_{LF} | \phi \rangle = M^2 \langle \zeta | \phi \rangle, \]
  with
  \[ \langle \zeta | H_{LF} | \phi \rangle = \left[ -\frac{d^2}{d\zeta^2} - \frac{1 - 4\nu^2}{4\zeta^2} \right] \langle \zeta | \phi \rangle, \]
  in the conformal limit.

• Holographic light-front wave functions \( \phi(\zeta) = \langle \zeta | \phi \rangle \) are normalized by
  \[ \langle \phi | \phi \rangle = \int d\zeta |\langle \zeta | \phi \rangle|^2 = 1, \]
  and represent the probability amplitude to find \( n \)-partons at transverse impact separation \( \zeta = z \). Its eigenmodes determine the hadronic mass spectrum.

• AdS/CFT equation as an effective light-front Schrödinger equation: relativistic, covariant and analytically tractable.
If \( \nu^2 > 0 \) the LF Hamiltonian, \( H_{LF} \), is written as a bilinear form

\[
H_{LF}^\nu(\zeta) = \Pi_\nu^\dagger(\zeta) \Pi_\nu(\zeta), \quad \nu^2 \geq 0,
\]

in terms of the operator

\[
\Pi_\nu(\zeta) = -i \left( \frac{d}{d\zeta} - \nu + \frac{1}{2} \right),
\]

and its adjoint

\[
\Pi_\nu^\dagger(\zeta) = -i \left( \frac{d}{d\zeta} + \nu + \frac{1}{2} \right),
\]

with commutation relations

\[
\left[ \Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta) \right] = \frac{2\nu + 1}{\zeta^2}.
\]

For \( \nu^2 \geq 0 \) the Hamiltonian is positive definite

\[
\langle \phi | H_{LF}^\nu | \phi \rangle = \int d\zeta |\Pi_\nu \phi(z)|^2 \geq 0
\]

and thus \( M^2 \geq 0 \).
For $\nu^2 < 0$ the Hamiltonian cannot be written as a bilinear form, but

$$\langle \phi | H'_{LF} | \phi \rangle \geq 2\nu^2 \int d\zeta \frac{\phi^2}{\zeta^2},$$

and the Hamiltonian is not bounded from below ("Fall-to-the-center" problem in Q.M.)

- Critical value of the potential corresponds to $\nu = 0$ with potential

$$V_{crit}(\zeta) = \frac{1}{4\zeta^2}.$$

- The Q.M. stability conditions are equivalent to the Breitenlohner-Freedman stability conditions

$$(\mu R)^2 \geq -\frac{d^4}{4}.$$  

For $d = 4$, $(\mu R)^2 = -4 + \nu^2$, and thus $\nu = 0$ correspond to the lowest stable solution.
Ladder Construction of Orbital States

• Orbital excitations constructed by the $L$-th application of the raising operator $a_L^\dagger = -i\Pi_L$ on the ground state:

\[ a_L^\dagger |L\rangle = c_L |L + 1\rangle. \]

• In the light-front $\zeta$-representation

\[ \phi_L(\zeta) = \langle \zeta | L \rangle = C_L \sqrt{\zeta} (-\zeta)^L \left( \frac{1}{\zeta} \frac{d}{d\zeta} \right)^L J_0(\zeta \mathcal{M}) \]

\[ = C_L \sqrt{\zeta} J_L (\zeta \mathcal{M}). \]

• The solutions $\phi_L$ are solutions of the light-front equation ($L = 0, \pm 1, \pm 2, \cdots$)

\[
\left[ -\frac{d^2}{d\zeta^2} - \frac{1 - L^2}{4\zeta^2} \right] \phi(\zeta) = \mathcal{M}^2 \phi(\zeta), \quad \nu = L
\]

• Mode spectrum from boundary conditions $\phi(\zeta = 1/\Lambda_{QCD}) = 0$, thus $\mathcal{M}^2 = \beta_{Lk}\Lambda_{QCD}$.

• The effective wave equation in the two-dim transverse LF plane has the Casimir representation $L^2$ corresponding to the $SO(2)$ rotation group [The Casimir for $SO(N) \sim S^{N-1}$ is $L(L + N - 2)$ ].
Fig: Orbital and radial AdS modes in the hard wall model for $\Lambda_{QCD} = 0.32$ GeV.
Higher Spin Bosonic Modes HW

• Each hadronic state of integer spin $S \leq 2$ is dual to a normalizable string mode

$$\Phi(x, z)_{\mu_1 \mu_2 \cdots \mu_S} = \epsilon_{\mu_1 \mu_2 \cdots \mu_S} e^{-iP \cdot x} \Phi_S(z).$$

with four-momentum $P_\mu$ and spin polarization indices along the 3+1 physical coordinates.

• Wave equation for spin $S$-mode


$$\left[ z^2 \partial_z^2 - (d+1-2S)z \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2 \right] \Phi_S(z) = 0,$$

• Solution

$$\tilde{\Phi}(z)_S = \left( \frac{z}{R} \right)^S \Phi(z)_S = Ce^{-iP \cdot x} z^{\frac{d}{2}} J_{\Delta - \frac{d}{2}}(z \mathcal{M}) \epsilon(P)_{\mu_1 \mu_2 \cdots \mu_S},$$

• We can identify the conformal dimension:

$$\Delta = \frac{1}{2} \left( d + \sqrt{(d - 2S)^2 + 4\mu^2 R^2} \right).$$

• Normalization:

$$R^{d-2S-1} \int_0^{\Lambda_{QCD}^{-1}} \frac{dz}{z^{d-2S-1}} \Phi^2_S(z) = 1.$$
• Upon the substitution $\phi(\zeta)_S \sim \zeta^{-3/2+S} \Phi(\zeta)_S$ in the spin-$S$ AdS wave equation ($d = 4$)

$$
\left[ - \frac{d^2}{d\zeta^2} - \frac{1 - 4\nu^2}{4\zeta^2} \right] \phi_S = M^2 \phi_S,
$$

where $(\mu R)^2 = -(2 - S)^2 + \nu^2$.

• Solution

$$
\phi(\zeta)_S = \epsilon_{\mu_1\mu_2\ldots\mu_S} \phi(\zeta),
$$

where the profile function $\phi(\zeta)$ is the solution for the scalar mode.

• Stable solutions satisfy a generalized B-F bound ($d = 4$)

$$(\mu R)^2 \geq -\frac{(d - 2S)^2}{4}.$$

• For the ground state $\Delta = 2$, independent of $S$. Higher excitations created by the ladder operators, thus $\nu = L$ and $\Delta = 2 + L$. Lowest stable solution corresponds to $L = 0$ for every spin mode $S$.

• AdS shifted field $\tilde{\Phi}$ couples to the interpolating operator $\mathcal{O}^S$ with scaling dimensions

$$
[\mathcal{O}^S] = d - [\tilde{\Phi}_S] = 2 + L.
$$
<table>
<thead>
<tr>
<th>State</th>
<th>$I$</th>
<th>$J^P$</th>
<th>$L$</th>
<th>$S$</th>
<th>$\mathcal{O}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(140)$</td>
<td>1</td>
<td>0$^-$</td>
<td>0</td>
<td>0</td>
<td>$\overline{q}\gamma_5 \frac{1}{2} \tau q$</td>
</tr>
<tr>
<td>$b_1(1235)$</td>
<td>1</td>
<td>1$^+$</td>
<td>1</td>
<td>0</td>
<td>$-i\overline{q}\gamma_5 \vec{\partial} \frac{1}{2} \tau q$</td>
</tr>
<tr>
<td>$\pi_2(1670)$</td>
<td>1</td>
<td>2$^+$</td>
<td>2</td>
<td>0</td>
<td>$-\overline{q}\gamma_5 \frac{1}{2} (3\partial_i \partial_j - \delta_{ij} \vec{\partial}^2) \frac{1}{2} \tau q$</td>
</tr>
<tr>
<td>$\ldots $</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(770)$</td>
<td>1</td>
<td>1$^-$</td>
<td>0</td>
<td>1</td>
<td>$q^\dagger \vec{\alpha} \frac{1}{2} \tau q$</td>
</tr>
<tr>
<td>$\omega(782)$</td>
<td>0</td>
<td>1$^-$</td>
<td>0</td>
<td>1</td>
<td>$q^\dagger \vec{\alpha} q$</td>
</tr>
<tr>
<td>$a_1(1260)$</td>
<td>1</td>
<td>1$^+$</td>
<td>1</td>
<td>1</td>
<td>$-iq^\dagger (\vec{\alpha} \times \vec{\partial}) \frac{1}{2} \tau q$</td>
</tr>
<tr>
<td>$f_2(1270)$</td>
<td>0</td>
<td>2$^+$</td>
<td>1</td>
<td>1</td>
<td>$-iq^\dagger [\frac{3}{2} (\alpha_i \partial_j + \alpha_j \partial_i) - \vec{\alpha} \cdot \vec{\partial} \delta_{ij}] q$</td>
</tr>
<tr>
<td>$f_1(1285)$</td>
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<td>1$^+$</td>
<td>1</td>
<td>1</td>
<td>$-iq^\dagger (\vec{\alpha} \times \vec{\partial}) q$</td>
</tr>
<tr>
<td>$a_2(1320)$</td>
<td>1</td>
<td>2$^+$</td>
<td>1</td>
<td>1</td>
<td>$-iq^\dagger [\frac{3}{2} (\alpha_i \partial_j + \alpha_j \partial_i) - \vec{\alpha} \cdot \vec{\partial} \delta_{ij}] \frac{1}{2} \tau q$</td>
</tr>
<tr>
<td>$a_0(1450)$</td>
<td>1</td>
<td>0$^+$</td>
<td>1</td>
<td>1</td>
<td>$-iq^\dagger \vec{\alpha} \cdot \vec{\partial} \frac{1}{2} \tau q$</td>
</tr>
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<td>$\ldots $</td>
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</table>

Tensor decomposition of total angular momentum interpolating operators $\mathcal{O}$, $[\mathcal{O}] = 2 + L$
Fig: Light meson and vector meson orbital spectrum $\Lambda_{QCD} = 0.32$ GeV
Non-Conformal Extension of Algebraic Integrability (SW Model)

- Soft-wall model [Karch, Katz, Son and Stephanov (2006)] retain conformal AdS metrics but introduce smooth cutoff which depends on the profile of a dilaton background field $\varphi(z) = \kappa^2 z^2$.

$$ S = \int d^4x \, dz \sqrt{g} \, e^{-\varphi(z)} \mathcal{L}. $$ (1)

- SW model can be constructed by extension of the conformal operator algebra. Consider the generator

$$ \Pi_\nu(\zeta) = -i \left( \frac{d}{d\zeta} - \nu + \frac{1}{2} - \kappa^2 \zeta \right), $$

and its adjoint

$$ \Pi_\nu^\dagger(\zeta) = -i \left( \frac{d}{d\zeta} + \nu + \frac{1}{2} + \kappa^2 \zeta \right), $$

with commutation relations

$$ \left[ \Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta) \right] = \frac{2\nu + 1}{\zeta^2} - 2\kappa^2. $$

- The LF Hamiltonian

$$ H_{LF} = \Pi_\nu^\dagger \Pi_\nu + C $$

is positive definite $\langle \phi | H_{LF} | \phi \rangle \geq 0$ for $\nu^2 \geq 0$, and $C \geq -4\kappa^2$. 
• Identify the zero mode \((C = -4\kappa^2)\) with the pion.

• Orbital and radial excited states are constructed from the ladder operators from the \(\nu = 0\) state.

• Light-front Hamiltonian equation

\[
H_{LF}|\phi\rangle = \mathcal{M}^2|\phi\rangle,
\]

leads to effective LF Schrödinger wave equation (KKSS)

\[
\left[-\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + \kappa^4 \zeta^2 + 2\kappa^2 (L - 1)\right]\phi(\zeta) = \mathcal{M}^2 \phi(\zeta)
\]

with eigenvalues \(\mathcal{M}^2 = 4\kappa^2 (n + L)\) and eigenfunctions

\[
\phi_L(\zeta) = \kappa^{1+L} \sqrt{\frac{2n!}{(n + L)!}} \zeta^{1/2 + L} e^{-\kappa^2 \zeta^2/2} L_n^L (\kappa^2 \zeta^2).
\]

• Transverse oscillator in the LF plane with \(SO(2)\) rotation subgroup has Casimir \(L^2\) representing rotations in the transverse LF plane.

• SW model is a remarkable example of integrability to a non-conformal extension of AdS/CFT

(Chim and Zamolodchikov (1992) - Potts Model.)
Fig: Orbital and radial AdS modes in the soft wall model for $\kappa = 0.6$ GeV.

Light meson orbital (a) and radial (b) spectrum for $\kappa = 0.6$ GeV.
Higher Spin Bosonic Modes SW

- Effective LF Schrödinger wave equation

\[
\left[-\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + \kappa^2 \zeta^2 + 2\kappa^2 (L + S - 1)\right] \phi_S(\zeta) = M^2 \phi_S(\zeta)
\]

with eigenvalues \(M^2 = 2\kappa^2 (2n + 2L + S)\).

- Compare with Nambu string result (rotating flux tube):

\[
M_n^2(L) = 2\pi\sigma (n + L + 1/2).
\]

- Glueballs in the bottom-up approach: (HW) Boschi-Filho, Braga and Carrion (2005); (SW) Colangelo, De Facio, Jugeau and Nicotri (2007).

Vector mesons orbital (a) and radial (b) spectrum for \(\kappa = 0.54\) GeV.
4 Fermionic Modes

- Baryons Spectrum in "bottom-up" holographic QCD

- Conformal metric $x^\ell = (x^\mu, z)$:
  \[
  ds^2 = g_{\ell m} dx^\ell dx^m = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2).
  \]

- Action for massive fermionic modes on AdS$_{d+1}$:
  \[
  S[\overline{\Psi}, \Psi] = \int d^{d+1}x \sqrt{g} \overline{\Psi}(x, z) \left( i \Gamma^\ell D_\ell - \mu \right) \Psi(x, z).
  \]

- Equation of motion:
  \[
  \left( i \Gamma^\ell D_\ell - \mu \right) \Psi(x, z) = 0
  \]
  \[
  \left[ i \left( z \eta^{\ell m} \Gamma_\ell \partial_m + \frac{d}{2} \Gamma_z \right) + \mu R \right] \Psi(x^\ell) = 0.
  \]
Holographic Light-Front Integrable Form and Spectrum

• In the conformal limit fermionic spin-$\frac{1}{2}$ modes $\psi(z)$ and spin-$\frac{3}{2}$ modes $\psi_\mu(z)$ are solutions of the Dirac light-front equation

$$H_{LF}|\psi\rangle = M|\psi\rangle,$$

with

$$H_{LF} = \alpha \Pi.$$

The operator

$$\Pi_\nu(\zeta) = -i \left( \frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta} \gamma_5 \right),$$

and its adjoint $\Pi_\nu^\dagger(\zeta)$ satisfy the commutation relations

$$\left[ \Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta) \right] = \frac{2\nu + 1}{\zeta^2} \gamma_5.$$

• Supersymmetric QM between bosonic and fermionic modes in AdS?
• Note: in the Weyl representation \((i\alpha = \gamma_5\beta)\)

\[
i\alpha = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]

• Baryon: twist-dimension \(3 + L\) \((\nu = L + 1)\)

\[
\mathcal{O}_{3+L} = \psi D_{\{\ell_1 \ldots \ell_q}\} \psi D_{\ell_{q+1} \ldots \ell_m}\} \psi, \quad L = \sum_{i=1}^{m} \ell_i.
\]

• Solution to Dirac eigenvalue equation with UV matching boundary conditions

\[
\psi(\zeta) = C \sqrt{\zeta} \left[ J_{L+1}(\zeta \mathcal{M})u_+ + J_{L+2}(\zeta \mathcal{M})u_- \right].
\]

Baryonic modes propagating in AdS space have two components: orbital \(L\) and \(L + 1\).

• Hadronic mass spectrum determined from IR boundary conditions

\[
\psi_{\pm} (\zeta = 1/\Lambda_{QCD}) = 0,
\]

given by

\[
\mathcal{M}_{\nu,k}^+ = \beta_{\nu,k} \Lambda_{QCD}, \quad \mathcal{M}_{\nu,k}^- = \beta_{\nu+1,k} \Lambda_{QCD},
\]

with a scale independent mass ratio.
<table>
<thead>
<tr>
<th>$SU(6)$</th>
<th>$S$</th>
<th>$L$</th>
<th>Baryon State</th>
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<td>56</td>
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<td>$N_{\frac{1}{2}}^+$ (939)</td>
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<tr>
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<td>$\Delta_{\frac{3}{2}}^+$ (1232)</td>
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<td></td>
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<td>$N_{\frac{1}{2}}^-$ (1650) $N_{\frac{3}{2}}^-$ (1700) $N_{\frac{5}{2}}^-$ (1675)</td>
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<tr>
<td></td>
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<td>$\Delta_{\frac{1}{2}}^+$ (1910) $\Delta_{\frac{3}{2}}^+$ (1920) $\Delta_{\frac{5}{2}}^+$ (1905) $\Delta_{\frac{7}{2}}^+$ (1950)</td>
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<td>$N_{\frac{7}{2}}^-$ $N_{\frac{9}{2}}^-$ $N_{\frac{11}{2}}^-$ $N_{\frac{13}{2}}^-$</td>
</tr>
</tbody>
</table>
Fig: Light baryon orbital spectrum for $\Lambda_{QCD} = 0.25$ GeV in the HW model. The 56 trajectory corresponds to $L$ even $P = +$ states, and the 70 to $L$ odd $P = -$ states.

- New analysis: $\Delta(1930)$ as a $L = 1$, $N = 1$, $J = \frac{3}{2}$ state. Horn et. al. arXiv: 0711.1138
We write the Dirac equation

\[ (\alpha \Pi(\zeta) - \mathcal{M}) \psi(\zeta) = 0, \]

in terms of the matrix-valued operator \( \Pi_\nu \),

\[ \Pi_\nu(\zeta) = -i \left( \frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta} \gamma_5 - \kappa^2 \zeta \gamma_5 \right), \]

Commutation relations for fermionic generators

\[ \left[ \Pi_\nu(\zeta), \Pi^\dagger_\nu(\zeta) \right] = \left( \frac{2\nu + 1}{\zeta^2} - 2\kappa^2 \right) \gamma_5. \]

Solutions to the Dirac equation

\[ \psi^+_\nu(\zeta) \sim z^{\frac{1}{2} + \nu} e^{-\kappa^2 \zeta^2 / 2} L_{n}^{\nu}(\kappa^2 \zeta^2), \]
\[ \psi^-_\nu(\zeta) \sim z^{\frac{3}{2} + \nu} e^{-\kappa^2 \zeta^2 / 2} L_{n+1}^{\nu+1}(\kappa^2 \zeta^2). \]

Eigenvalues

\[ \mathcal{M}^2 = 4\kappa^2 (n + \nu + 1). \]
Linear Holographic Confinement

- Compare with usual Dirac equation in AdS space \((x^\ell = (x^\mu, z))\)

\[
\left[ i \left( \eta^\ell_m \Gamma_\ell \partial_m + \frac{d}{2} \Gamma_z \right) + \mu R + V(z) \right] \Psi(x^\ell) = 0.
\]

in presence of a linear confining potential \(V(z) = \kappa^2 z\).

- Upon substitution \(\Psi(x, z) = e^{-iP \cdot x} z^2 \psi(z), \ z \to \zeta\) we find

\[
\alpha \Pi(\zeta) \psi(\zeta) = \mathcal{M} \psi(\zeta)
\]

with

\[
\Pi_\nu(\zeta) = -i \left( \frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta} \gamma_5 - \kappa^2 \zeta \gamma_5 \right), \quad \mu R = \nu + \frac{1}{2},
\]

our previous result.

- Soft-wall model for baryons corresponds to a linear confining potential in the LF transverse variable \(\zeta\)!
• Baryon: twist-dimension $3 + L \quad (\nu = L + 1)$

$$\mathcal{O}_{3+L} = \psi \{D_{\ell_1} \ldots D_{\ell_q} \psi D_{\ell_{q+1}} \ldots D_{\ell_m}\} \psi, \quad L = \sum_{i=1}^{m} \ell_i.$$ 

• Define the zero point energy (identical as in the meson case) $\mathcal{M}^2 \rightarrow \mathcal{M}^2 - 4\kappa^2$:

$$\mathcal{M}^2 = 4\kappa^2(n + L + 1).$$

Fig: Proton Regge Trajectory $\kappa = 0.49$ GeV
Excitation Spectrum of Baryons

- Hard wall holographic model: $\mathcal{M}_n(L) \sim L + 2n$
- Soft wall holographic model: $\mathcal{M}^2_n(L) \sim L + n$
- Quark models (shell structure of excitations): $\mathcal{M}^2_n(L) \sim L + 2n$
- Observed same multiplicity of states for mesons and baryons!
5 Current Matrix Elements in AdS Space: The Form Factor

- Hadronic matrix element for EM coupling with string mode $\Phi(x^\ell)$, $x^\ell = (x^\mu, z)$

$$ig_5 \int d^4x \, dz \, \sqrt{g} A^\ell(x, z) \Phi^*_P(x, z) \overleftarrow{\partial}^\ell \Phi_P(x, z).$$

- Electromagnetic probe polarized along Minkowski coordinates ($Q^2 = -q^2 > 0$)

$$A(x, z)_\mu = \epsilon_\mu e^{-iQ \cdot x} J(Q, z), \quad A_z = 0.$$  

- Propagation of external current inside AdS space described by the AdS wave equation

$$\left[ z^2 \partial_z^2 - z \partial_z - z^2 Q^2 \right] J(Q, z) = 0,$$

subject to boundary conditions $J(Q = 0, z) = J(Q, z = 0) = 1$.

- Solution

$$J(Q, z) = z Q K_1(zQ).$$

- Substitute hadronic modes $\Phi(x, z)$ in the AdS EM matrix element

$$\Phi_P(x, z) = e^{-iP \cdot x} \Phi(z), \quad \Phi(z) \rightarrow z^\Delta, \quad z \rightarrow 0.$$
• Find form factor in AdS as overlap of normalizable modes dual to the in and out hadrons \( \Phi_P \) and \( \Phi^{P'} \), with the non-normalizable mode \( J(Q, z) \) dual to external source \([\text{Polchinski and Strassler (2002)}]\).

\[
F(Q^2) = R^3 \int_0^{\Lambda_{\text{QCD}}^{-1}} \frac{dz}{z^3} \Phi(z) J(Q, z) \Phi(z).
\]

• Since \( K_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \), the external source is suppressed inside AdS for large \( Q \). Important contribution to the integral is from \( z \sim 1/Q \), where \( \Phi \sim z^\Delta \).

• For large \( Q^2 \)

\[
F(Q^2) \rightarrow \left[ \frac{1}{Q^2} \right]^{\Delta-1},
\]

and the power-law ultraviolet point-like scaling is recovered \([\text{Polchinski and Susskind (2001)}]\).

Fig: Suppression of external modes for large \( Q \) inside AdS. Red curves: \( J(Q, z) \), black: \( \Phi(z) \).
Current Propagation in the SW Model

- Propagation of external current inside AdS space described by the AdS wave equation

\[
[z^2 \partial_z^2 - z (1 + 2\kappa^2 z^2) \partial_z - Q^2 z^2] J_\kappa(Q, z) = 0.
\]

- Solution: bulk-to-boundary propagator

\[
J_\kappa(Q, z) = \Gamma\left(1 + \frac{Q^2}{4\kappa^2}\right) U\left(\frac{Q^2}{4\kappa^2}, 0, \kappa^2 z^2\right),
\]

where \( U(a, b, c) \) is the confluent hypergeometric function

\[
\Gamma(a)U(a, b, z) = \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{b-a-1} dt.
\]

- Form factor in presence of the dilaton background \( \varphi = \kappa^2 z^2 \)

\[
F(Q^2) = R^3 \int \frac{dz}{z^3} e^{-\kappa^2 z^2} \Phi(z) J_\kappa(Q, z) \Phi(z).
\]

- For large \( Q^2 \gg 4\kappa^2 \)

\[
J_\kappa(Q, z) \to zQ K_1(zQ) = J(Q, z),
\]

the external current decouples from the dilaton field.
Space and Time-Like Pion Form Factor

• Hadronic string modes $\Phi_\pi(z) \to z^2$ as $z \to 0$ (twist $\tau = 2$)

\[
\Phi^{HW}_\pi(z) = \frac{\sqrt{2} \Lambda_{QCD}}{R^{3/2} J_1(\beta_{0,1})} z^2 J_0(z \beta_{0,1} \Lambda_{QCD}),
\]

\[
\Phi^{SW}_\pi(z) = \frac{\sqrt{2} \kappa}{R^{3/2}} z^2.
\]

• $F_\pi$ has analytical solution in the SW model

\[
F_\pi(Q^2) = \frac{4 \kappa^2}{4 \kappa^2 + Q^2}.
\]

Fig: $F_\pi(q^2)$ for $\kappa = 0.375$ GeV and $\Lambda_{QCD} = 0.22$ GeV. Continuous line: SW, dashed line: HW.
• Scaling behavior for large $Q^2$: $Q^2 F_\pi(Q^2) \to \text{constant}$  \textbf{Pion $\tau = 2$}

Fig: Continuous line: SW model for $\kappa = 0.375$ GeV. Dashed line: HW model for $\Lambda_{QCD} = 0.22$ GeV.
• Analytical continuation to time-like region $q^2 \to -q^2$ ($M_\rho = 4\kappa^2 = 750$ MeV)

• Strongly coupled semiclassical gauge/gravity limit hadrons have zero widths (stable).

Vector Mesons: Hong, Yoon and Strassler (2004); Grigoryan and Radyushkin (2007).

Space and time-like pion form factor for $\kappa = 0.375$ GeV in the SW model.
• Extract the value of the mean pion charge radius

\[ F_\pi(Q^2) = 1 - \frac{1}{6} \langle r^2_\pi \rangle Q^2 + \mathcal{O}(Q^4), \quad \langle r^2_\pi \rangle = -6 \frac{dF_\pi(Q^2)}{dQ^2} \bigg|_{Q^2=0}. \]

• Soft wall model

\[ \langle r^2_\pi \rangle_{SW} = \frac{3}{2 \kappa^2} \simeq 0.42 \text{ fm}^2, \]

compared with the PDG value \( \langle r^2_\pi \rangle = 0.45(1) \text{ fm}^2. \)

• Hard-wall model with non-confined electromagnetic current expand \( J(Q^2, z) \) for small values of \( Q^2 \)

\[ J(Q^2, z) = 1 + \frac{z^2 Q^2}{4} \left[ 2 \gamma - 1 + \ln \left( \frac{z^2 Q^2}{4} \right) \right] + \mathcal{O}(4), \]

where \( \gamma = 0.5772 \ldots \) Since there is no scale in \( J(Q^2, z) \), value of \( \langle r^2_\pi \rangle \) diverges logarithmically.

• Problem in defining \( \langle r^2_\pi \rangle \) does not appear if one uses Neumann boundary conditions for the HW model:

\[ \langle r^2_\pi \rangle_{HW} \sim 1/\Lambda^2_{QCD}. \]

(Grigoryan and Radyushkin (2007))
Note: Analytical Form of Hadronic Form Factor for Arbitrary Twist

- Form factor for a string mode with scaling dimension $\tau$, $\Phi_\tau$ in the SW model

$$F(Q^2) = \Gamma(\tau) \frac{\Gamma\left(1 + \frac{Q^2}{4\kappa^2}\right)}{\Gamma\left(\tau + \frac{Q^2}{4\kappa^2}\right)}.$$ 

- For $\tau = N$, $\Gamma(N + z) = (N - 1 + z)(N - 2 + z) \ldots (1 + z)\Gamma(1 + z)$.

- Form factor expressed as $N - 1$ product of poles

$$F(Q^2) = \frac{1}{1 + \frac{Q^2}{4\kappa^2}}, \quad N = 2,$$

$$F(Q^2) = \frac{2}{\left(1 + \frac{Q^2}{4\kappa^2}\right)\left(2 + \frac{Q^2}{4\kappa^2}\right)}, \quad N = 3,$$

$$\ldots$$

$$F(Q^2) = \frac{(N - 1)!}{\left(1 + \frac{Q^2}{4\kappa^2}\right)\left(2 + \frac{Q^2}{4\kappa^2}\right)\ldots\left(N - 1 + \frac{Q^2}{4\kappa^2}\right)}, \quad N.$$ 

- For large $Q^2$:

$$F(Q^2) \to (N - 1)! \left[\frac{4\kappa^2}{Q^2}\right]^{(N-1)}.$$
Consider the spin non-flip form factors

\[
F_+(Q^2) = g_+ \int d\zeta J(Q, \zeta) |\psi_+(\zeta)|^2,
\]

\[
F_-(Q^2) = g_- \int d\zeta J(Q, \zeta) |\psi_-(\zeta)|^2,
\]

where the effective charges \(g_+\) and \(g_-\) are determined from the spin-flavor structure of the theory.

Choose the struck quark to have \(S^z = +1/2\). The two AdS solutions \(\psi_+(\zeta)\) and \(\psi_-(\zeta)\) correspond to nucleons with \(J^z = +1/2\) and \(-1/2\).

For \(SU(6)\) spin-flavor symmetry

\[
F^p_1(Q^2) = \int d\zeta J(Q, \zeta) |\psi_+(\zeta)|^2,
\]

\[
F^n_1(Q^2) = -\frac{1}{3} \int d\zeta J(Q, \zeta) \left[ |\psi_+(\zeta)|^2 - |\psi_-(\zeta)|^2 \right],
\]

where \(F^p_1(0) = 1\), \(F^n_1(0) = 0\).
• Scaling behavior for large $Q^2$: $Q^4 F_1^p(Q^2) \rightarrow \text{constant}$  

Proton $\tau = 3$

• Scaling behavior for large $Q^2$: $Q^4 F^p_1(Q^2) \rightarrow \text{constant}$

\[ \text{Neutron } \tau = 3 \]

6 Light Front Dynamics

- Different possibilities to parametrize space-time in terms of general coordinates $\bar{x}(x)$ (excluding all related by a Lorentz transformation).

- According to Dirac there are no more than three different parametrization of space-time, the *instant form*, the *front form* and the *point form*, Dirac (1949).

- Parametrizations differ by the hypersurface on which the initial conditions are specified. Each evolve with different “times” and has its own Hamiltonian, but should give the same physical results.

- *Instant form*: hypersurface defined by $t = 0$, the familiar one.

- *Front form*: hypersurface is tangent to the light cone.

- *Point form*: hypersurface is an hyperboloid
The instant form

\( \tilde{x}^0 = ct \)
\( \tilde{x}^1 = x \)
\( \tilde{x}^2 = y \)
\( \tilde{x}^3 = z \)

\[ \tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

The front form

\( \tilde{x}^0 = ct + z \)
\( \tilde{x}^1 = x \)
\( \tilde{x}^2 = y \)
\( \tilde{x}^3 = ct - z \)

\[ \tilde{g}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \]

The point form

\( \tilde{x}^0 = \tau \), \( ct = \tau \cosh \omega \)
\( \tilde{x}^1 = \omega \), \( x = \tau \sinh \omega \sin \theta \cos \phi \)
\( \tilde{x}^2 = \theta \), \( y = \tau \sinh \omega \sin \theta \sin \phi \)
\( \tilde{x}^3 = \phi \), \( z = \tau \sinh \omega \cos \theta \)

\[ \tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tau^2 & 0 & 0 \\ 0 & 0 & -\tau^2 \sin^2 \omega & 0 \\ 0 & 0 & 0 & -\tau^2 \sin^2 \omega \sin^2 \theta \end{pmatrix} \]
Light-Front Fock Representation

• Light-front expansion constructed by quantizing QCD at fixed light-cone time \( \tau = t + z/c \) and forming the invariant light-front Hamiltonian (Brodsky, Pauli and Pinski, Phys. Rept. 301 299 (1998)):

\[
H_{LF} = P^+ P^- - \vec{P}^2
\]

where \( P^\pm = P^0 \pm P^z \).

• Momentum generators \( P^+ \) and \( \vec{P} \) are kinematical (independent of the interactions) and \( P^- = i \frac{d}{d\tau} \) generates light-front time translations.

• Eigenvalues of \( H_{LF} \) give the mass spectrum of the color-singlet hadron states:

\[
H_{LF} |\psi_h\rangle = M_h^2 |\psi_h\rangle.
\]

• State \( |\psi_h\rangle \) is an expansion in multi-particle Fock eigenstates \( |n\rangle \) of the free light-front Hamiltonian:

\[
|\psi_h\rangle = \sum_n \psi_{n/h} |n\rangle.
\]

• Example: \( |P\rangle = |uud\rangle + |uudg\rangle + |uud\bar{q}q\rangle \ldots \)
• Coefficients of the Fock expansion $\psi_{n/h}$ are independent of the total momentum $P^+$ and $P_\perp$ of the hadron and depend only on the relative partonic coordinates of parton $i$ in Fock-state $n$: momentum fraction $x_i = k_i^+ / P^+$ and $k_\perp i$

$$\sum_{i=1}^{n} x_i = 1 \quad \sum_{i=1}^{n} k_{\perp i} = 0.$$ 

• Fock components

$$\psi_{n/h}(x_i, k_{\perp i}) = \langle n; x_i, k_{\perp i} | \psi_h(P^+, P_\perp) \rangle,$$

frame independent and encode hadron properties in high momentum-transfer collisions.
Current Matrix Elements in the QCD Light-Front Frame

- Electromagnetic form factor \( (P' = P + q) \)

\[
\langle P' | J^+(0) | P \rangle = 2 (P + P')^+ F(Q^2).
\]

- Drell-Yan-West (DYW) expression for meson form factor integrated over phase-space momentum

\[
F(q^2) = \sum_n \int [dx_i] \int [d^2 k_{\perp i}] \sum_j e_j \psi^*_{n/P'}(x_i, k'_{\perp i}, \lambda_i) \psi_{n/P}(x_i, k_{\perp i}, \lambda_i),
\]

where \( k'_{\perp i} = k_{\perp i} + (1 - x_i) q_{\perp} \) for a struck quark and \( k'_{\perp i} = k_{\perp i} - x_i q_{\perp} \) for each spectator. The formula is exact if the sum is over all Fock states \( n \).

- Normalization of LFWFs

\[
\sum_n \int [dx_i] \int [d^2 k_{\perp i}] |\psi_{n/h}(x_i, k_{\perp i})|^2 = 1,
\]
• Transverse position coordinates $x_i r_{\perp i} = x_i R_{\perp} + b_{\perp i}$

$$\sum_{i=1}^{n} b_{\perp i} = 0, \quad \sum_{i=1}^{n} x_i r_{\perp i} = R_{\perp}. $$

• LFWF $\psi_n(x_j, k_{\perp j})$ expanded in terms of $n-1$ independent coordinates $b_{\perp j}, j = 1, 2, \ldots, n-1$

$$\psi_n(x_j, k_{\perp j}) = (4\pi)^{n-1} \frac{1}{2} \prod_{j=1}^{n-1} \int d^2 b_{\perp j} \exp \left( i \sum_{j=1}^{n-1} b_{\perp j} \cdot k_{\perp j} \right) \tilde{\psi}_n(x_j, b_{\perp j}).$$

• Normalization

$$\sum_{n} \prod_{j=1}^{n-1} \int dx_j d^2 b_{\perp j} |\tilde{\psi}_n(x_j, b_{\perp j})|^2 = 1.$$

• The form factor has the exact representation (DYW)

$$F(q^2) = \sum_{n} \prod_{j=1}^{n-1} \int dx_j d^2 b_{\perp j} \exp \left( i q_{\perp} \cdot \sum_{j=1}^{n-1} x_j b_{\perp j} \right) |\tilde{\psi}_n(x_j, b_{\perp j})|^2,$$

corresponding to a change of transverse momentum $x_j q_{\perp}$ for each of the $n-1$ spectators and elementary coupling to the struck parton.
• Define effective single particle transverse density (Soper '77)

\[ F(q^2) = \int_0^1 dx \rho(x, \vec{q}_\perp) \]

with

\[ \rho(x, \vec{q}_\perp) = \int d^2 \vec{\eta}_\perp e^{i\vec{\eta}_\perp \cdot \vec{q}_\perp} \tilde{\rho}(x, \vec{\eta}_\perp). \]

• From DYW expression for FF in transverse position space:

\[ \tilde{\rho}(x, \vec{\eta}_\perp) = \sum_{n} \prod_{j=1}^{n-1} \int dx_j d^2 b_{\perp j} \delta(1 - x - \sum_{j=1}^{n-1} x_j) \delta^{(2)}(\sum_{j=1}^{n-1} x_j b_{\perp j} - \vec{\eta}_\perp) \left| \tilde{\psi}_n(x_j, b_{\perp j}) \right|^2 \]

• Integration over the \( n - 1 \) spectator partons, and \( x = x_n \) is the coordinate of the active quark.

• \( \vec{\eta}_\perp = \sum_{j=1}^{n-1} x_j b_{\perp j} \) is the \( x \)-weighted transverse position coordinate of the \( n - 1 \) spectators.
Holographic Mapping of AdS Modes to QCD LFWFs

• Integrate Soper formula over angles:

\[
F(q^2) = 2\pi \int_0^1 dx \frac{(1 - x)}{x} \int \zeta d\zeta J_0 \left( \zeta q \sqrt{\frac{1 - x}{x}} \right) \tilde{\rho}(x, \zeta),
\]

with \( \tilde{\rho}(x, \zeta) \) QCD effective transverse charge density.

• Transversality variable

\[
\zeta = \sqrt{\frac{x}{1-x}} \left| \sum_{j=1}^{n-1} x_j b_{\perp j} \right|.
\]

• Compare AdS and QCD expressions of FFs for arbitrary \( Q \) using identity:

\[
\int_0^1 dx J_0 \left( \zeta Q \sqrt{\frac{1 - x}{x}} \right) = \zeta Q K_1(\zeta Q),
\]

the solution for \( J(Q, \zeta) = \zeta Q K_1(\zeta Q) \)!
• Hadronic QCD transverse density $\tilde{\rho}$ is identified with the string mode density $|\Phi|^2$ in AdS space!

$$
\tilde{\rho}(x, \zeta) = \frac{R^3}{2\pi} \frac{x}{1 - x} \frac{|\Phi(\zeta)|^2}{\zeta^4}
$$

SJB and GdT (2006)

• Transverse variable $\zeta$ represents the invariant separation between point-like constituents and is also the holographic variable: $\zeta = z$.

• For two-partons $\zeta^2 = x(1 - x)b_\perp^2$

$$
\tilde{\rho}(x, \zeta) = \frac{1}{(1 - x)^2} \left| \tilde{\psi}(x, \zeta) \right|^2.
$$

• Two-parton holographic bound state LFWF

$$
\left| \tilde{\psi}(x, \zeta) \right|^2 = \frac{R^3}{2\pi} x(1 - x) \frac{|\Phi(\zeta)|^2}{\zeta^4}.
$$

• $\text{AdS}_3$ dual to $1 + 1$ large $N_C$ QCD (t’Hooft Model). Mapping between parton-$x$ and radial $\text{AdS}_3$ coordinate: Katz and Okui (2007).
Example: Pion LFWF

- Two parton LFWF bound state:

\[ \tilde{\psi}^{HW}_{\bar{q}q/\pi}(x, b_{\perp}) = \frac{\Lambda_{\text{QCD}}}{\sqrt{\pi}} \frac{\sqrt{x(1-x)}}{J_{1+L}(\beta_{L,k})} J_L \left( \sqrt{x(1-x)} |b_{\perp}| \beta_{L,k} \Lambda_{\text{QCD}} \right) \theta \left( b_{\perp}^2 \leq \frac{\Lambda_{\text{QCD}}^2}{x(1-x)} \right), \]

\[ \tilde{\psi}^{SW}_{\bar{q}q/\pi}(x, b_{\perp}) = \frac{\kappa^{L+1}}{\sqrt{\pi}} \sqrt{\frac{n!}{(n+L)!}} \left[ x(1-x) \right]^{\frac{1}{2}+L} |b_{\perp}|^L e^{-\frac{1}{2} \kappa^2 x(1-x)} b_{\perp}^2 L_n^L \left( \kappa^2 x(1-x) b_{\perp}^2 \right). \]

Fig: Ground state pion LFWF in impact space. (a) HW model \( \Lambda_{\text{QCD}} = 0.32 \text{ GeV} \), (b) SW model \( \kappa = 0.375 \text{ GeV} \).
Example: Evaluation of QCD Matrix Elements

- Pion decay constant $f_\pi$ defined by the matrix element of EW current $J^+_W$:

$$
\langle 0 | \bar{\psi}_u \gamma^+ \frac{1}{2} (1 - \gamma_5) \psi_d | \pi^- \rangle = i \frac{P^+ f_\pi}{\sqrt{2}}
$$

with

$$
| \pi^- \rangle = |d\bar{u}\rangle = \frac{1}{\sqrt{N_C}} \frac{1}{\sqrt{2}} \sum_{c=1}^{N_C} \left( b^\dagger_c d_c u^\uparrow - b^\dagger_c d_c u^\downarrow \right) |0\rangle.
$$

- Find light-front expression (Lepage and Brodsky '80):

$$
f_\pi = 2 \sqrt{N_C} \int_0^1 dx \int \frac{d^2 \vec{k}_\perp}{16 \pi^3} \psi_{\bar{q}q/\pi}(x, k_\perp).
$$

- Using relation between AdS modes and QCD LFWF in the $\zeta \rightarrow 0$ limit

$$
f_\pi = \frac{1}{8} \sqrt{\frac{3}{2}} R^{3/2} \lim_{\zeta \rightarrow 0} \frac{\Phi(\zeta)}{\zeta^2}.
$$

- Holographic result ($\Lambda_{QCD} = 0.22$ GeV and $\kappa = 0.375$ GeV from pion FF data): Exp: $f_\pi = 92.4$ MeV

$$
f_{HW}^\pi = \frac{\sqrt{3}}{8 J_1(\beta_0, k)} \Lambda_{QCD} = 91.7$ MeV, \quad f_{SW}^\pi = \frac{\sqrt{3}}{8} \kappa = 81.2$ MeV,
$$
Perturbative CFT vs Non-Perturbative AdS Results

- Heavy quark potential: \( g_s N_C \rightarrow \sqrt{g_s N_C} \) Maldacena (1998), Rey and Yee (1998).

- Distribution amplitude \( \phi_M(x, Q) \sim \int Q^2 d^2k_\perp \psi_{\bar{q}q/M}(x, k_\perp) \).

- Second Moment of the Distribution \( (\xi = 1 - 2x) \)
  \[
  \langle \xi^2 \rangle = \frac{1}{5} \quad \phi_{\text{PQCD}} \sim x(1 - x),
  \]
  \[
  \langle \xi^2 \rangle = \frac{1}{4} \quad \phi_{\text{AdS/QCD}} \sim \sqrt{x(1 - x)}.
  \]

- Sachrajda lattice result: \( \langle \xi^2 \rangle = 0.28 \pm 0.02 \)
Introduction of Heavy Quark Masses

- Introduction of light quark masses in AdS/QCD involve complex dynamics, as the evolution from current to constituent quark masses should be included.

- Assume the momentum space LFWF is a function of the invariant (off-energy shell)

\[ \mathcal{M}^2 - \mathcal{E} = \sum_{i=1}^{n} \left( k_{\perp i}^2 + m_i^2 \right) / x_i. \]

- Soft-Wall LFWF ansatz for bound state with massive constituents:

\[ \psi(x, k_{\perp}) \sim \frac{4\pi}{\kappa \sqrt{x(1-x)}} e^{-\frac{1}{2\kappa^2} \left( \frac{k_{\perp}^2}{x(1-x)} + \frac{m_1^2}{x} + \frac{m_2^2}{1-x} \right)}, \]

- Fourier transform to impact space

\[ \widetilde{\psi}(x, b_{\perp}) \sim \frac{\kappa}{\sqrt{\pi}} \sqrt{x(1-x)} e^{-\frac{1}{2\kappa^2} \kappa^2 x(1-x)b_{\perp}^2 + \frac{1}{2\kappa^2} \left[ \frac{m_1^2}{x} - \frac{m_2^2}{1-x} \right]} \cdot \]

\[ z \rightarrow \zeta \xrightarrow{m_i} \chi \]

\[ \chi^2 = x(1-x)b_{\perp}^2 + \frac{1}{\kappa^4} \left[ \frac{m_1^2}{x} + \frac{m_2^2}{1-x} \right]. \]
Fig: LFWF in impact space: $\psi_{J/\Psi}(x, b)$, $\psi_D(x, b)$

- $J/\Psi$ meson: $m_1 = m_2 = m_c$;
- $D$ meson: $m_1 = 0$, $m_2 = m_c$, $m_c = 1.25$ GeV.